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A nonlocal inhomogeneous dispersal process ☆

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Abstract

This article in devoted to the study of the nonlocal dispersal equation

$$u_t(x,t) = \int\limits_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)}{g(y)}\,dy - u(x,t) \quad \text{in } \mathbb{R}\times[0,\infty),$$

and its stationary counterpart. We prove global existence for the initial value problem, and under suitable hypothesis on g and J, we prove that positive bounded stationary solutions exist. We also analyze the asymptotic behavior of the finite mass solutions as $t\to\infty$, showing that they converge locally to zero.

Keywords: Integral equation; Nonlocal dispersal; Inhomogeneous dispersal

1. Introduction

Let $K:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ be a nonnegative smooth function such that $\int_{\mathbb{R}^N} K(x,y)\,dx = 1$ for all $y\in\mathbb{R}^N$. Equations of the form

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$$u_t(x,t) = \int_{\mathbb{R}^N} K(x,y)u(y,t)\,dy - u(x,t), \tag{1.1}$$

have been widely used to model diffusion processes in the following sense. As stated in [9,10] if $u(y,t)$ is thought of as a density at location y at time t and $K(x,y)$ as the probability distribution of jumping from location y to location x, then the rate at which individuals from all other places are arriving to location x is

$$\int_{\mathbb{R}^N} K(x,y)u(y,t)\,dy.$$

On the other hand, the rate at which individuals are leaving location x to travel to all other places is

$$-\int_{\mathbb{R}^N} K(y,x)u(x,t)\,dy = -u(x,t).$$

In the absence of external sources this implies that the density u must satisfy equation (1.1).

A more specific dispersal model that has been treated by several authors in different contexts, is the case when K is a convolution kernel. More precisely they consider

$$K(x,y) = J(x-y),$$

where $J:\mathbb{R}^N \to \mathbb{R}$ is a nonnegative function such that $\int_{\mathbb{R}^N} J(y)\,dy = 1$. See for example [1,2,4,7,8] for the study of travelling waves, [3,11] for asymptotic behavior and [5] and [6] for the case of bounded domains. If in the above model we assume that the support of J is the unit ball of $\mathbb{R}^N$ centered at the origin, we have that individuals at location x are not allowed to jump, up to probability 0, off the unit ball centered at x. We will say in such a case that we are dealing with a process of step size one.

The purpose of this paper is to study the one-dimensional spatial case with kernels of the form

$$K(x,y) = J\left(\frac{x-y}{g(y)}\right)\frac{1}{g(y)}.$$

In this case the dispersal is inhomogeneous and the step size, $g(y)$, of the dispersal depends on the position y. Therefore in this paper we will deal with the following problem:

$$u_t(x,t) = \int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)}{g(y)}\,dy - u(x,t) \quad \text{in } \mathbb{R}\times[0,\infty), \tag{1.2}$$

with a prescribed initial data

$$u(x,0) = u_0(x) \quad \text{on } \mathbb{R}.$$

An important role in the study of the behavior of solutions of (1.2) is played by the solutions of the corresponding stationary problem, namely

$$p(x)=\int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\,dy \quad \text{in } \mathbb{R}. \tag{1.3}$$

The existence and properties of solutions of problems (1.2) and (1.3) depend strongly on the function g, specially in the case that g vanishes at some places. Actually the dependence is rather on how g vanishes than on the plain fact that it vanishes.

Throughout all of this paper we will make the following assumptions on J and g.

The function $J:\mathbb{R}\to\mathbb{R}$ will be a nonnegative, smooth, even function with $\int_{\mathbb{R}} J(r)\,dr=1$. We shall assume also that the support of J is $[-1,1]$ which means $J(x)>0$ if and only if $x\in(-1,1)$.

For the function g we assume:

(g1) $g:\mathbb{R}\to\mathbb{R}$ is continuous and $0\leqslant g\leqslant b<\infty$ in $\mathbb{R}$.

(g2) The set $\{x\in\mathbb{R}\mid g(x)=0\}\cap[-K,K]$ is finite for any $K>0$. If $g(\bar{x})=0$ then there exist $r>0$, $C>0$ and $0<\alpha<1$ such that $g(x)\geqslant C|x-\bar{x}|^{\alpha}$ for all $x\in[\bar{x}-r,\bar{x}+r]$.

Under these basic hypotheses we prove that (1.2) has a globally defined mass preserving solution for any given $u_0\in L^1$. Moreover even though g can vanish at some points, these solutions have an infinite speed of propagation in the sense that if $u_0\geqslant 0$ and $u_0\neq 0$, then $u(x,0)>0$ for all x and all $t>0$.

In order to study the asymptotic behavior of solutions of (1.2) we are lead to the analysis of Eq. (1.3). In this direction we seek nonnegative solutions that play the role of the constant solutions when $g\equiv C$. We will prove, under a slightly strengthened version of (g2), the existence of bounded positive solutions that are also bounded away from 0. These stationary solutions permit us to define, following ideas of [12], a Lyapunov's functional that allow us to prove the local convergence to zero of solutions of (1.2).

Solutions of (1.3) will be obtained as the limit as $K\to\infty$ of solutions of the following stationary problem:

$$\int_{-K}^{K} J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\,dy=\int_{-K}^{K} J\left(\frac{x-y}{g(x)}\right)\frac{p(x)}{g(x)}\,dy, \quad x\in[-K,K]. \tag{1.4}$$

A key tool in the passage to the limit is the surprising fact that if p is a bounded solution of (1.3) then the quantity

$$W(x)=\int_0^b\int_{x-w}^{x+w} p(s)\int_{\frac{w}{g(s)}}^{1} J(z)\,dz\,ds\,dw$$

is constant. This identity implies a Harnack's type inequality which provides some estimates needed in the proof.

For the sake of completeness we also study the corresponding evolution Neumann problem namely, for $x \in [-K, K]$ and $t \geqslant 0$ we consider

$$\begin{cases} u_t(x,t) = \displaystyle\int_{-K}^{K} J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)}{g(y)}\,dy - \int_{-K}^{K} J\left(\frac{x-y}{g(x)}\right)\frac{u(x,t)}{g(x)}\,dy, \\ u(0,x) = u_0(x), \end{cases} \tag{1.5}$$

and its relation with (1.4).

Problem (1.5) can be regarded as an homogeneous Neumann problem in the sense that the flow of individuals through the boundary is null and hence the integral $\int_{-K}^{K} u(y,t)\,dy$ remains constant in time. In this fashion, problem (1.4) can be thought of as a stationary Neumann problem.

We should mention that the results we are obtaining, such as the infinite speed of propagation and the existence of bounded steady states, are strongly dependent on the vanishing profile of g, which is expressed in hypothesis (g2). For example, if we change (g2) by $g(x) \leqslant C|x-\bar{x}|^{\alpha}$ with $\alpha > 1$, then the existence of a barrier prevents an infinite speed of propagation. We will pursue the study of (1.2) with g with this profile in a future work.

This paper is organized as follows. Section 2 is devoted to the Neumann type problems, that is (1.5) and (1.4). In Section 3 we study problem (1.2). Problem (1.3) is studied in Section 4 and in Section 5 we deal with the asymptotic behavior of solutions of (1.2).

2. The Neumann problem

We note that for $x \in [-K, K]$ and $t \geqslant 0$, problem (1.5) can be written as

$$\begin{cases} u_t(x,t) = (T_0 u)(x,t) - \alpha(x)u(x,t), \\ u(x,0) = u_0(x), \end{cases} \tag{2.1}$$

where

$$T_0 u(x) = \int_{-K}^{K} J\left(\frac{x-y}{g(y)}\right)\frac{u(y)}{g(y)}\,dy,$$

and

$$\alpha(x) = \begin{cases} \int_{-K}^{K} J(\frac{x-y}{g(x)})\frac{1}{g(x)}\,dy & \text{if } g(x) \neq 0, \\ 1 & \text{if } g(x) = 0 \text{ and } x \neq -K, K, \\ \frac{1}{2} & \text{if } g(x) = 0 \text{ and } x = -K \text{ or } x = K. \end{cases}$$

It is easy to check that there exists $c_0 > 0$ such that $\alpha(x) > c_0$ for all $x \in [-K, K]$ and, according to our assumptions, α is continuous in $[-K, K]$. Moreover by (g2)

$$\int_{-K}^{K} \frac{1}{g(y)}\,dy < \infty. \tag{2.2}$$

For existence and uniqueness of solutions of (2.1) we have the following theorem whose proof is standard and will be only sketched.

Theorem 2.1. *Given $u_0 \in L^1[-K,K]$ there exists a unique solution $u \in C^1(\mathbb{R}, L^1[-K,K])$ of* (2.1). *The solution u is mass preserving, that is*

$$\int_{-K}^{K} u(x,t)\,dx = \int_{-K}^{K} u_0(x)\,dx,$$

for all $t > 0$. Moreover, if $u_0 \in C([-K,K])$ then $u \in C^1(\mathbb{R}, C([-K,K]))$.

Proof. The operator T_0 maps $L^1[-K,K]$ into $L^1[-K,K]$ and is continuous. Thus, by standard semigroup theory (see [13, Theorem 1.2]), for any $u_0 \in L^1[-K,K]$ the initial value problem has a unique solution $u \in C^1(\mathbb{R}, L^1[-K,K])$ which satisfies the following integral equation

$$u(x,t) = u(x,t_0)e^{\alpha(x)(t_0-t)} + \int_{t_0}^{t} e^{\alpha(x)(s-t)} \int_{-K}^{K} J\left(\frac{x-y}{g(y)}\right)\frac{u(y,s)}{g(y)}\,dy\,ds \tag{2.3}$$

a.e., for all $t_0 \leqslant t$. The fact that the integral is preserved follows by integration in the equation and the last statement about continuity is a consequence of (2.3). □

Our next result shows that, even if g vanishes at some points, hypothesis (g2) guarantees that the process has infinite speed of propagation.

Proposition 2.1. *If $u_0 \in L^1[-K,K]$ is nonnegative a.e. in $[-K,K]$, then $u(x,t) \geqslant 0$ a.e. in $[-K,K]$ for each $t \geqslant 0$. If in addition $u_0 \not\equiv 0$, then $u(x,t) > 0$ a.e. in $[-K,K]$ for all $t > 0$.*

Proof. To prove that $u(x,t)$ is nonnegative we observe that, according to (2.3), for a small interval $[0,t]$ the solution u can be obtained as the unique fixed point of a map which leaves invariant the positive cone in $L^1[-K,K]$.

Suppose now that $u_0 \geqslant 0$ a.e. with $u_0 > 0$ in a set of positive measure. Observe that by (2.3) if $u(x,s) > 0$ in $E_0 \subset [-K,K]$ with $|E_0| > 0$ then $u(x,t) > 0$ in E_0 for $t \geqslant s$.

Let $x_1 < \cdots < x_N$ be the ordered set of zeroes of g in $[-K,K]$. We set $r > 0$, $0 < \alpha < 1$, $C > 0$ such that $g(x) \geqslant C|x - \bar{x}|^\alpha$ for all $x \in [x_i - r, x_i + r]$. Redefining $r > 0$ if necessary in (g2), we assume that $C(r/2)^\alpha > 3r$ and that $|\{x \in Z \mid u_0(x) > 0\}| > 0$, where $Z = [-K,K] \setminus \bigcup_{i=1}^N (x_i - r, x_i + r)$. We denote $\delta = \min\{g(y) \mid y \in [-K,K] \setminus \bigcup_{i=1}^N (x_i - r/2, x_i + r/2)\}$.

Consider an interval $I = [\tilde{a}, \tilde{b}] \subset Z$ such that $0 < \tilde{b} - \tilde{a} \leqslant \delta/2$, $|I \cap \{x \in [-K,K] \mid u_0(x) > 0\}| > 0$ and $I \subset [x_i + r, x_{i+1} - r]$ for some i. By the definition of δ we have that

$$\int_{-K}^{K} J\left(\frac{x-y}{g(y)}\right)\frac{u_0(y)}{g(y)}\,dy > 0 \quad \text{a.e. in } [\tilde{a} - \delta/2, \tilde{b} + \delta/2],$$

thus from (2.3) we have that $u(t,x) > 0$ a.e. in $[\tilde{a} - \delta/2, \tilde{b} + \delta/2]$ for all $t > 0$. Repeating this argument, we obtain that $u(t,x) > 0$ a.e. in $[x_i + r/2, x_{i+1} - r/2]$ for all $t > 0$. Observe that if $y \in [x_i + r/2, x_i + r]$ then $g(y) \geqslant C|y - x_i|^\alpha \geqslant 3r$. Then if $x_i - r \leqslant x \leqslant x_{i+1} + r$ we have that

$$\int_{-K}^{K} J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)}{g(y)}\,dy > 0 \quad \text{a.e. for } t > 0,$$

thus $u(t,x) > 0$ a.e. in $[x_i - 2r, x_{i+1} + 2r]$ for all $t > 0$. Iterating the above procedure we obtain the desired result. $\square$

Remark 2.1. If $u_0 \in C([-K,K])$ is nonnegative and nontrivial, then $u(x,t) > 0$ for all $t > 0$ and $x \in [-K,K]$.

In order to study the asymptotic behavior as $t \to \infty$ of the positive solutions of (1.5), we will first establish the existence of a positive continuous steady state, that is a solution of (1.4). This existence result will be a consequence of Krein–Rutman's theorem, see [14], applied to the operator $T : C([-K,K]) \to C([-K,K])$ defined by

$$Tu(x) = \frac{1}{\alpha(x)} T_0 u(x).$$

The next lemmas will be used in the proof. The first one states the strong positivity of T and its proof, which is similar to the one of Proposition 2.1, will be omitted.

Lemma 2.1. *Let $u \in C([-K,K])$ be such that $u \geqslant 0$ and $u_0 \neq 0$. Then there exists $n \in \mathbb{N}$ such that $(T^n u)(x) > 0$ in $[-K,K]$.*

Lemma 2.2. *The family*

$$\left\{T_0 f(x) \;\middle|\; f : [-K,K] \to \mathbb{R},\ \|f\|_\infty \leqslant 1\right\}$$

is equicontinuous.

Proof. Let $\varepsilon > 0$. By condition (g2) we get that there exists $\delta > 0$ such that

$$\int_{\substack{[-K,K]\\ g(y)<\delta}} \frac{1}{g(y)}\,dy < \frac{\varepsilon}{4\|J\|_\infty}.$$

Since J is uniformly continuous in $[-1,1]$ there exists $\eta > 0$ such that if $|w - \bar{w}| < \eta/\delta$ then $|J(w) - J(\bar{w})| < \varepsilon\delta/2(b-a)$. Then, if $|x - z| < \eta$ we have that

$$\left|T_0 f(x) - T_0 f(z)\right| \leqslant 2 \int_{\substack{[-K,K]\\ g(y)<\delta}} \frac{\|J\|_\infty}{g(y)}\,dy + \frac{1}{\delta} \int_{\substack{[-K,K]\\ g(y)\geqslant\delta}} \left|J\left(\frac{x-y}{g(y)}\right) - J\left(\frac{z-y}{g(y)}\right)\right| dy \leqslant \varepsilon,$$

hence, $T_0(B(0,1))$ is equicontinuous. $\square$

As a consequence of this lemma we have

Lemma 2.3. $T : C([-K,K]) \to C([-K,K])$ *is a compact operator.*

Now we are ready to give our existence result for steady states.

Theorem 2.2. *There exists a unique positive solution* u^* *of* (1.4) *with* $\int_{-K}^{K} u^*\,dx = 1$.

Proof. Lemmas 2.1 and 2.3 guarantee, via Krein–Rutman's theorem, that there exists $\lambda > 0$ and a unique positive solution u^* of $Tu^* = \lambda u^*$, with $\int_{-K}^{K} u^*\,dx = 1$. Note that u^* satisfies

$$\lambda \int_{-K}^{K} \alpha(x)u(x)\,dx = \int_{-K}^{K}\int_{-K}^{K} J\left(\frac{x-y}{g(y)}\right)\frac{u(y)}{g(y)}\,dy\,dx = \int_{-K}^{K} \alpha(y)u(y)\,dy,$$

hence $\lambda = 1$ and u^* is the desired solution. $\square$

We will study the asymptotic behavior of the solutions of (1.5) as $t \to \infty$. We start with the case $u_0 \in C([-K,K])$.

Theorem 2.3. *For any* $K > 0$ *there exists* $\gamma > 0$ *such that if* $u_0 \in C([-K,K])$, $u_0 \geqslant 0$ *and* $\int_{-K}^{K} u_0(x)\,dx = C$, *then the solution* $u(x,t)$ *of* (1.5) *with initial condition* $u_0(x)$ *satisfies*

$$\left\| u(\cdot,t) - Cu^*(\cdot)\right\|_\infty \leqslant e^{-\gamma t}\|u_0 - Cu^*\|_\infty \quad \textit{for } t > 0.$$

Proof. Let $v_0 \in C([-K,K])$ with $\int_{-K}^{K} v_0\,dx = 0$ and denote $v(x,t)$ the solution of (1.5) with initial data v_0. By direct integration in the equation of (1.5) we obtain that $\int_{-K}^{K} v(t,x)\,dx = \int_{-K}^{K} v_0\,dx = 0$ for all $t > 0$. Set $X = \{f \in C([-K,K]) \mid \int_{-K}^{K} f\,dx = 0\}$, then $T_0 - \alpha(x)I : X \to X$, and by standard semigroup theory our result will be proved if we show that the spectrum $\sigma_X(T_0 - \alpha(x)I)$ is contained in the open half-plane $\{\operatorname{Re} z < 0\}$. Suppose that $\mu = \tilde\alpha + i\tilde\beta$, with $\tilde\alpha \geqslant 0$, belongs to $\sigma_X(T_0 - \alpha(x)I)$. By Fredholm's Alternative theorem μ is an eigenvalue, thus there exists a nontrivial $v \in X$ such that $T_0 v - \alpha(x)v = \mu v$. Using Krein–Rutman's theorem we obtain that $\mu \neq 0$, since 1 is a simple eigenvalue of T with positive eigenfunction.

Let $w = w_1 + iw_2 \in X$ be an eigenfunction associated to μ. Then for some $\gamma > 0$ we have that $\gamma u^* + w_1 \geqslant 0$ in $[-K,K]$, $\gamma u^* + w_1 \not\equiv 0$ and $\gamma u^*(x_0) + w_1(x_0) = 0$ for some $x_0 \in [-K,K]$. Set $u(t)$ the solution of (1.5) with initial value $\gamma u^* + w_1$ which is given by $u(t) = \gamma u^* + e^{\tilde\alpha t}\operatorname{Re}(e^{i\tilde\beta t}w)$. If $\tilde\alpha > 0$, then for large $t > 0$ we have that there exists $x \in [-K,K]$ such that $u(x,t) < 0$ contradicting Proposition 2.1. When $\tilde\alpha = 0$ we have that $u(x_0, \frac{2\pi}{\tilde\beta}) = 0$ which also contradicts Proposition 2.1. $\square$

In the case $u_0 \in L^1[-K,K]$ with $u_0 \geqslant 0$ a.e. the asymptotic behavior of $u(\cdot,t)$ is a consequence of Theorem 2.3 and the following lemma.

Lemma 2.4. *Let* u *be a solution of* (1.5), *then*

$$\left\| u(\cdot,t)\right\|_{L^1[-K,K]} \leqslant \|u_0\|_{L^1[-K,K]} \quad \textit{for all } t \geqslant 0.$$

Proof. Write $u_0 = h_0^+ - h_0^-$ where $h_0^+ = \max(u_0, 0)$ and $h_0^- = \min(u_0, 0)$ and let h^+ and h^- be the solutions of (1.5) with initial conditions h_0^+ and h_0^-, respectively. Now by linearity we have

$$u(x,t) = h^+(x,t) - h^-(x,t).$$

Hence

$$\left|u(x,t)\right| \leqslant h^+(x,t) + h^-(x,t),$$

and then

$$\begin{aligned}\int_{-K}^{K} \left|u(x,t)\right| dx &\leqslant \int_{-K}^{K} h^+(x,t)\,dx + \int_{-K}^{K} h^-(x,t)\,dx \\ &= \int_{-K}^{K} h_0^+(x)\,dx + \int_{-K}^{K} h_0^-(x)\,dx = \int_{-K}^{K} \left|u_0(x)\right| dx. \qquad \square\end{aligned}$$

Theorem 2.4. *Let $u_0 \in L^1(-K,K)$ with $u_0 \geqslant 0$ a.e. and let $u(x,t)$ be the solution of* (1.5) *with initial data u_0, then*

$$\left\|u(\cdot,t) - Cu^*(\cdot)\right\|_{L^1[-K,K]} \to 0,$$

as $t \to \infty$, where $C = \int_{-K}^{K} u_0(x)\,dx$.

Proof. Let $\varepsilon > 0$. Pick $u_0^\varepsilon \in C([-K,K])$ such that $u_0^\varepsilon \geqslant 0$, $\int_{-K}^{K} u_0^\varepsilon(x)\,dx = C$ and $\|u_0 - u_0^\varepsilon\|_{L^1[-K,K]} \leqslant \varepsilon$. Let u^ε be the solution of (1.5) with initial condition u_0^ε. One has

$$\left\|u(\cdot,t) - Cu^*(\cdot)\right\|_{L^1} \leqslant \left\|u(\cdot,t) - u^\varepsilon(\cdot)\right\|_{L^1} + \left\|u^\varepsilon(\cdot,t) - Cu^*(\cdot)\right\|_{L^1},$$

and the proposition follows from Lemma 2.4 and Theorem 2.3. $\square$

3. The Cauchy problem

In this section we establish some basic facts about solutions of (1.2). We start by defining the operator

$$\mathcal{L}u(x,t) = \int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)}{g(y)}\,dy.$$

Proposition 3.1. *The operator $\mathcal{L}$ maps continuously $L^1(\mathbb{R})$ into $L^1(\mathbb{R})$ and*

$$\int_{\mathbb{R}} \mathcal{L}v = \int_{\mathbb{R}} v.$$

Moreover, if in addition

(g3) *there is* $a_3 > 0$ *and* $0 < \beta < 1$ *such that*

$$\int\limits_{g(y)<\alpha} J\left(\frac{x-y}{g(y)}\right)\frac{1}{g(y)}\,dy < \beta,$$

holds, then $\mathcal{L}$ *is a continuous map from* $L^\infty(\mathbb{R})$ *to* $L^\infty(\mathbb{R})$.

Proof. Observe that by Fubini's theorem we have

$$\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{v(y)}{g(y)}\,dy\,dx = \int\limits_{\mathbb{R}} v(y)\,dy.$$

This implies that $\mathcal{L}$ is a continuous operator from $L^1(\mathbb{R})$ to $L^1(\mathbb{R})$ and $\int_{\mathbb{R}} \mathcal{L}v = \int_{\mathbb{R}} v$. Now, if $v \in L^\infty$ and (g3) holds we have

$$\begin{aligned}\left|\mathcal{L}v(x)\right| &\leqslant \int\limits_{g(y)<a_3} J\left(\frac{x-y}{g(y)}\right)\frac{|v(y)|}{g(y)}\,dy + \int\limits_{g(y)\geqslant a_3} J\left(\frac{x-y}{g(y)}\right)\frac{|v(y)|}{g(y)}\,dy \\ &\leqslant \|v\|_\infty \beta + \frac{2b\|J\|_\infty}{a_3}\|v\|_\infty. \qquad \square\end{aligned}$$

Theorem 3.1. *Given* $u_0 \in L^1(\mathbb{R})$ *there exists a unique solution* $u \in C^1(\mathbb{R}, L^1(\mathbb{R}))$ *of* (1.2). *The solution* u *conserves the total mass, that is*

$$\int\limits_{\mathbb{R}} u(x,t)\,dx = \int\limits_{\mathbb{R}} u_0(x)\,dx$$

for all $t > 0$. *Moreover, if* (g3) *holds and* $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ *then* $u(\cdot,t) \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$.

Proof. The result is a direct consequence of Proposition 3.1 and standard results from semigroup theory. $\square$

It is convenient at this point to introduce the integral form of the initial value problem (1.2),

$$u(x,t) = u(x,t_0)e^{-(t-t_0)} + \int\limits_{t_0}^{t} e^{-(t-s)} \int\limits_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{u(y,s)}{g(y)}\,dy\,ds. \tag{3.1}$$

Our next result states that this problem has infinite speed of propagation.

Proposition 3.2. *If* $u_0 \in L^1(\mathbb{R})$ *is nonnegative a.e., then* $u(x,t) \geqslant 0$ *a.e. in* $\mathbb{R}$ *for each* $t \geqslant 0$. *If in addition* $u_0 \not\equiv 0$, *then* $u(x,t) > 0$ *a.e. in* $\mathbb{R}$ *for all* $t > 0$.

Proof. It follows by the same arguments as the proof of Proposition 2.1. $\square$

Proposition 3.3. *Suppose that* (g3) *holds and* $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, *then*

(i) *there exists a constant* $K_\infty(u_0)$ *such that for all* $t \in \mathbb{R}^+$,

$$\big\|u(\cdot,t)\big\|_\infty \leqslant K_\infty;$$

(ii) *for any* $p \geqslant 1$, *there exists a constant* $K_p(u_0)$, *such that* $\|u\|_{L^p(\mathbb{R})} \leqslant K_p$;

(iii) $u(x,t)$ *is globally Lipschitz in time, uniformly in space that is, there exists a constant* $\kappa(u_0)$ *such that*

$$\big\|u(\cdot,t) - u(\cdot,s)\big\|_\infty \leqslant \kappa|t-s|,$$

for all $t, s \geqslant 0$.

Proof. Suppose that for some sequence $t_n \to \infty$ we have $\|u(\cdot,t_n)\|_\infty \to \infty$. Then we can find a sequence $T_n \to \infty$ such that $\|u(\cdot,T_n)\|_\infty \to \infty$ and

$$\sup_{0\leqslant t\leqslant T_n} \big\|u(\cdot,t)\big\|_\infty = \big\|u(\cdot,T_n)\big\|_\infty. \tag{3.2}$$

Observe that the solution u satisfies

$$\big(e^t u\big)_t(x,t) = e^t \int\limits_{g(y)<a_3} J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)}{g(y)}\,dy + e^t \int\limits_{g(y)\geqslant a_3} J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)}{g(y)}\,dy. \tag{3.3}$$

Integrating this equality between 0 and T_n, and using (3.2) we obtain

$$\big|u(x,T_n)\big| \leqslant e^{-T_n}\|u_0\|_\infty + \beta\big\|u(\cdot,T_n)\big\|_\infty + \frac{1}{a_3}\|J\|_\infty\|u_0\|_{L^1} \quad \text{a.e.},$$

which contradicts the fact that $\|u(\cdot,T_n)\|_\infty \to \infty$ and proves (i).

Since $\|u(\cdot,t)\|_\infty \leqslant K_\infty$ and (1.2) is mass preserving (ii) follows easily by interpolation.

To prove (iii) we integrate Eq. (1.2) to obtain

$$\begin{aligned}\big|u(x,t_2) - u(x,t_1)\big| &\leqslant \int_{t_1}^{t_2}\Bigg[\int\limits_{g(y)<a_3} J\left(\frac{x-y}{g(y)}\right)\frac{K_\infty}{g(y)}\,dy + \int\limits_{g(y)\geqslant a_3} J\left(\frac{x-y}{g(y)}\right)\frac{K_\infty}{g(y)}\,dy + K_\infty\Bigg]dt \\ &\leqslant |t_1 - t_2|K_\infty\left(\beta + \frac{\|J\|_\infty}{a_3} + 1\right),\end{aligned}$$

which concludes the proof. □

4. Steady states for the whole real line

In this section we will establish the existence of positive solutions of (1.3). As a first step we will construct a bounded positive solution of (1.3) under the extra assumption that g is constant near infinity.

Lemma 4.1. *Assume there exist $N > 0$ and positive constants c_1 and c_2 such that $g(x) \equiv c_1$ if $x \geqslant N$ and $g(x) \equiv c_2$ if $x \leqslant -N$. Then* (1.3) *has a nontrivial bounded solution.*

Proof. We will obtain the solution as the limit of a sequence of solutions of problem (1.4) as $K \to \infty$. To do this fix $K > N + b$, where b is an upper bound for the function g, and let p_K be a solution of (1.5) in the interval $[-K, K]$ normalized such that $\|p_K\|_\infty = 1$.

We claim that each $p_K : [-K, K] \to \mathbb{R}$ attains its maximum in the sub interval $[-(N + b), N + b]$. Indeed, let $x_0 \in [-K, K]$ be such that

$$p_K(x_0) = \max_{x\in[-K,K]} p_K(x) = 1.$$

Assume that $x_0 \in [N + b, K]$ and consider the set

$$A = \bigl\{x \in [N + b, K] \bigm| p_K(x) = 1\bigr\}.$$

The set A is clearly closed. On the other hand if $x_1 \in A$ one has

$$p_K(x_1) = \frac{1}{H(x_1)} \int_{-K}^{K} J\left(\frac{x_1 - y}{c_1}\right) \frac{p_K(y)}{c_1}\, dy, \tag{4.1}$$

where

$$H(x_1) = \int_{-K}^{K} J\left(\frac{x_1 - y}{c_1}\right) \frac{1}{c_1}\, dy.$$

Since the operator on the right-hand side of (4.1) is an average operator we obtain that $p_K(y) = 1$ for all $y \in [x_1 - c_1, x_1 + c_1] \cap [N + b, K]$. Hence A is also open in $[N + b, K]$. Since it is not empty we have $A = [N + b, K]$. In particular $M = p_K(N + b)$ and the maximum is also attained at $[-(N + b), N + b]$. A similar argument proves that if the maximum is attained at a point in $[-K, -(N + b)]$ then it is also attained at the point $-(N + b)$. Hence we have proved that p_K always attains its maximum in the sub interval $[-(N + b), N + b]$ as desired.

Arguing as in Lemma 2.2, the family p_K is equicontinuous in any fixed bounded interval. Thus, using Ascoli–Arzela's theorem and a standard diagonal procedure we can construct a sequence K_n with $K_n \to \infty$ as $n \to \infty$ and such that p_{K_n} converges uniformly, to a continuous function p, in compact subsets of $\mathbb{R}$ as $n \to \infty$. It is clear that p is a nonnegative solution of (1.3).

Finally since $p_{K_n}(x_{K_n}) = 1$ for some $x_{K_n} \in [-(N + b), N + b]$ it follows that p is nontrivial. $\square$

The following lemma, that will be used later, is of interest on itself.

Lemma 4.2. *For any bounded solution of p of* (1.3) *one has*

$$\int_0^b \int_{D-w}^{D+w} p(s) \int_{\frac{w}{g(s)}}^1 J(z)\,dz\,ds\,dw = \int_0^b \int_{C-w}^{C+w} p(s) \int_{\frac{w}{g(s)}}^1 J(z)\,dz\,ds\,dw$$

for any $C, D \in \mathbb{R}$.

Proof. Let p be a bounded solution of (1.3). Pick M and N such that

$$M + 2b \leqslant N.$$

Integrating (1.3) we get

$$\begin{aligned}
\int_M^N p(x)\,dx &= \int_M^N \int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\,dy\,dx = \int_{M-b}^{N+b} \int_M^N J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\,dx\,dy \\
&= \int_{M-b}^{M+b} \int_M^N J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\,dx\,dy + \int_{M+b}^{N-b} \int_M^N J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\,dx\,dy \\
&\quad + \int_{N-b}^{N+b} \int_M^N J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\,dx\,dy.
\end{aligned}$$

But since $g \leqslant b$ and $\int_{\mathbb{R}} J(z)\,dz = 1$ one has

$$\int_{M+b}^{N-b} \int_M^N J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\,dx\,dy = \int_{M+b}^{N-b} p(y)\,dy$$

and hence

$$\begin{aligned}
&\int_M^{M+b} p(x)\,dx + \int_{N-b}^N p(x)\,dx \\
&\quad = \int_{M-b}^{M+b} \int_M^N J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\,dx\,dy + \int_{N-b}^{N+b} \int_M^N J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\,dx\,dy.
\end{aligned}$$

Making, for fixed y, the change of variables $z = \frac{x-y}{g(y)}$ and using the fact that $M + 2b \leqslant N$ we have

$$\int_{N-b}^{N+b}\int_{M}^{N} J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\,dx\,dy = \int_{N-b}^{N+b} p(y)\int_{\frac{M-y}{g(y)}}^{\frac{N-y}{g(y)}} J(z)\,dz\,dy = \int_{N-b}^{N+b} p(y)\int_{-1}^{\frac{N-y}{g(y)}} J(z)\,dz\,dy,$$

which can be written as

$$\int_{N-b}^{N+b}\int_{M}^{N} J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\,dx\,dy = \int_{N-b}^{N} p(y)\int_{-1}^{\frac{N-y}{g(y)}} J(z)\,dz\,dy + \int_{N}^{N+b} p(y)\int_{-1}^{\frac{N-y}{g(y)}} J(z)\,dz\,dy.$$

Similarly we have

$$\int_{M-b}^{M+b}\int_{M}^{N} J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\,dx\,dy = \int_{M-b}^{M} p(y)\int_{\frac{M-y}{g(y)}}^{1} J(z)\,dz\,dy + \int_{M}^{M+b} p(y)\int_{\frac{M-y}{g(y)}}^{1} J(z)\,dz\,dy.$$

Setting

$$A_M = \int_{M}^{M+b} p(y)\,dy - \int_{M-b}^{M} p(y)\int_{\frac{M-y}{g(y)}}^{1} J(z)\,dz\,dy - \int_{M}^{M+b} p(y)\int_{\frac{M-y}{g(y)}}^{1} J(z)\,dz\,dy, \tag{4.2}$$

and

$$B_N = -\int_{N-b}^{N} p(y)\,dy + \int_{N-b}^{N} p(y)\int_{-1}^{\frac{N-y}{g(y)}} J(z)\,dz\,dy + \int_{N}^{N+b} p(y)\int_{-1}^{\frac{N-y}{g(y)}} J(z)\,dz\,dy,$$

we have that $A_M = B_N$ provided that $M + 2b \leqslant N$. This implies $A_M = B_N \equiv K$ for all $M, N \in \mathbb{R}$.

Let $C, D \in \mathbb{R}$ with $C < D$. Integrating (4.2) with respect to M from C to D we have

$$\begin{aligned}(D-C)K = {} & \int_{C}^{D}\int_{M}^{M+b} p(y)\,dy\,dM - \int_{C}^{D}\int_{M-b}^{M} p(y)\int_{\frac{M-y}{g(y)}}^{1} J(z)\,dz\,dy\,dM \\ & - \int_{C}^{D}\int_{M}^{M+b} p(y)\int_{\frac{M-y}{g(y)}}^{1} J(z)\,dz\,dy\,dM.\end{aligned} \tag{4.3}$$

But

$$\begin{aligned}\int_C^D \int_M^{M+b} p(y)\,dy\,dM &= \int_0^b \int_C^D p(w+M)\,dM\,dw \\ &= \int_0^b \int_C^{D-2w} p(w+M)\,dM\,dw + \int_0^b \int_{D-2w}^{D} p(w+M)\,dM\,dw \end{aligned} \tag{4.4}$$

also

$$\begin{aligned}\int_C^D \int_M^{M+b} p(y) \int_{\frac{M-y}{g(y)}}^{1} J(z)\,dz\,dy\,dM &= \int_C^D \int_0^b p(M+w) \int_{\frac{-w}{g(M+s)}}^{1} J(z)\,dz\,dw\,dM \\ &= \int_0^b \int_C^D p(M+w) \int_{\frac{-w}{g(M+w)}}^{1} J(z)\,dz\,dM\,dw \\ &= \int_0^b \int_C^{D-2w} p(M+w) \int_{\frac{-w}{g(M+w)}}^{1} J(z)\,dz\,dM\,dw \\ &\quad + \int_0^b \int_{D-2w}^{D} p(M+w) \int_{\frac{-w}{g(M+w)}}^{1} J(z)\,dz\,dM\,dw \end{aligned} \tag{4.5}$$

and

$$\begin{aligned}\int_C^D \int_{M-b}^{M} p(y) \int_{\frac{M-y}{g(y)}}^{1} J(z)\,dz\,dy\,dM &= \int_C^D \int_0^b p(M-b+s) \int_{\frac{b-s}{g(M-b+s)}}^{1} J(z)\,dz\,ds\,dM \\ &= \int_0^b \int_C^D p(M-b+s) \int_{\frac{b-s}{g(M-b+s)}}^{1} J(z)\,dz\,dM\,ds \\ &= \int_0^b \int_C^D p(M-w) \int_{\frac{w}{g(M-w)}}^{1} J(z)\,dz\,dM\,dw \\ &= \int_0^b \int_{C-2w}^{D-2w} p(R+w) \int_{\frac{w}{g(R+w)}}^{1} J(z)\,dz\,dR\,dw \end{aligned}$$

$$= \int_0^b \int_{C-2w}^{C} p(R+w) \int_{\frac{w}{g(R+w)}}^{1} J(z)\,dz\,dR\,dw$$
$$+ \int_0^b \int_{C}^{D-2w} p(R+w) \int_{\frac{w}{g(R+w)}}^{1} J(z)\,dz\,dR\,dw. \tag{4.6}$$

Since by the symmetry of J one has

$$\int_{\frac{-w}{g(M+w)}}^{1} J(z)\,dz + \int_{\frac{w}{g(M+w)}}^{1} J(z)\,dz = 1,$$

substituting the result of (4.4)–(4.6) in (4.3) one gets

$$(D-C)K = \int_0^b \int_{D-2w}^{D} p(M+w) \int_{-1}^{\frac{-w}{g(M+w)}} J(z)\,dz\,dM\,dw$$
$$- \int_0^b \int_{C-2w}^{C} p(M+w) \int_{\frac{w}{g(M+w)}}^{1} J(z)\,dz\,dM\,dw. \tag{4.7}$$

Because we have assumed p bounded, the right-hand side of (4.7) is bounded independently of the choice of C and D. This implies $K=0$ and hence

$$\int_0^b \int_{D-2w}^{D} p(M+w) \int_{-1}^{\frac{-w}{g(M+w)}} J(z)\,dz\,dM\,dw = \int_0^b \int_{C-2w}^{C} p(M+w) \int_{\frac{w}{g(M+w)}}^{1} J(z)\,dz\,dM\,dw \tag{4.8}$$

for all C and D provided that $C \leqslant D$. Or, what is the same,

$$\int_0^b \int_{D-w}^{D+w} p(s) \int_{-1}^{\frac{-w}{g(s)}} J(z)\,dz\,ds\,dw = \int_0^b \int_{C-w}^{C+w} p(s) \int_{\frac{w}{g(s)}}^{1} J(z)\,dz\,ds\,dw \tag{4.9}$$

for all C and D provided that $C \leqslant D$.

The lemma follows since the symmetry of J implies that

$$\int_{-1}^{\frac{-w}{g(s)}} J(z)\,dz = \int_{\frac{w}{g(s)}}^{1} J(z)\,dz. \qquad \square$$

As a consequence of Lemma 4.2 we have the following Harnack's type inequality.

Lemma 4.3. *Let p be a nonnegative bounded solution of* (1.3) *and $M > 0$ be so that $g(-M) \neq 0$ and $g(M) \neq 0$. Then there exists a constant $A > 0$, depending on M, J, b and g in $[-M-b, M+b]$, such that for any $x \in [-M, M]$ and any $D \in \mathbb{R}$ we have*

$$p(x) \leqslant A \int_{D-b}^{D+b} p(s)\, ds.$$

Proof. During this proof A will denote a constant depending on M, J and b that can change from step to step.

Let $x_0 \in [-M, M]$ be such that

$$p(x_0) = \max_{x \in [-M,M]} p(x).$$

For a fixed a such that $0 < a < b$ define

$$Z = \bigl\{ y \in [-M-b, M+b] \bigm| g(y) < a \text{ and } |x_0 - y| \leqslant g(y) \bigr\}$$

and

$$W = \bigl\{ y \in [-M-b, M+b] \bigm| g(y) \geqslant a \text{ and } |x_0 - y| \leqslant g(y) \bigr\}.$$

Then

$$p(x_0) = \int_Z J\left(\frac{x_0 - y}{g(y)}\right) \frac{p(y)}{g(y)}\, dy + \int_W J\left(\frac{x_0 - y}{g(y)}\right) \frac{p(y)}{g(y)}\, dy.$$

Since $g(-M) \neq 0$ and $g(M) \neq 0$ we can make a smaller if necessary to guarantee that

$$Z \subset [-M, M].$$

In this case we have

$$p(x_0) \leqslant p(x_0) \int_Z J\left(\frac{x_0 - y}{g(y)}\right) \frac{1}{g(y)}\, dy + \frac{\|J\|_\infty}{a} \int_W p(y)\, dy,$$

and, according to our hypotheses on g, we can take a smaller if necessary to have the existence of $\bar{\beta} < 1$ such that

$$\int_Z J\left(\frac{x_0 - y}{g(y)}\right) \frac{1}{g(y)}\, dy < \bar{\beta}.$$

So

$$(1-\bar{\beta})\,p(x_0) \leqslant \frac{\|J\|_\infty}{a} \int_W p(y)\,dy$$

from where

$$p(x_0) \leqslant A \int_W p(y)\,dy. \tag{4.10}$$

Now fix $x_1 \in [-M, M]$. Using Lemma 4.2 for any $C \in [-M, M]$ we obtain

$$\begin{aligned}
\int_0^b \int_{C-w}^{C+w} p(s) \int_{\frac{w}{g(s)}}^1 J(z)\,dz\,ds\,dw &\geqslant \int_{\frac{a}{4}}^{\frac{a}{2}} \int_{x_1-w}^{x_1+w} p(s) \int_{\frac{w}{g(s)}}^1 J(z)\,dz\,ds\,dw \\
&\geqslant \int_{\frac{a}{4}}^{\frac{a}{2}} \int_{[x_1-w,\,x_1+w]\cap W} p(s) \int_{\frac{w}{g(s)}}^1 J(z)\,dz\,ds\,dw \\
&\geqslant \int_{\frac{a}{4}}^{\frac{a}{2}} \int_{[x_1-w,\,x_1+w]\cap W} p(s) \int_{\frac{1}{2}}^1 J(z)\,dz\,ds\,dw \\
&\geqslant A \int_{\frac{a}{4}}^{\frac{a}{2}} \int_{[x_1-w,\,x_1+w]\cap W} p(s)\,ds\,dw \\
&\geqslant A\frac{a}{4} \int_{[x_1-\frac{a}{4},\,x_1+\frac{a}{4}]\cap W} p(s)\,ds.
\end{aligned} \tag{4.11}$$

Observe that there exists an integer N, depending on a and b, such that W can be covered by N intervals of length $\frac{a}{2}$ in the form

$$W \subset \bigcup_{i=1}^N \left[x_i - \frac{a}{4}, x_i + \frac{a}{4}\right].$$

This fact implies the existence of A such that

$$\int_W p(s)\,ds \leqslant A \int_0^b \int_{C-w}^{C+w} p(s) \int_{\frac{w}{g(s)}}^1 J(z)\,dz\,ds\,dw, \tag{4.12}$$

and using (4.10) we have

$$p(x_0) \leqslant A \int_0^b \int_{C-w}^{C+w} p(s) \int_{\frac{w}{g(s)}}^1 J(z)\,dz\,ds\,dw. \tag{4.13}$$

On the other hand,

$$\int_0^b \int_{C-w}^{C+w} p(s) \int_{\frac{w}{g(s)}}^1 J(z)\,dz\,ds\,dw \leqslant \int_0^b \int_{C-w}^{C+w} p(s)\,ds\,dw \leqslant b \int_{C-b}^{C+b} p(s)\,ds \tag{4.14}$$

which together with (4.13) proves the lemma. □

We are now in a position to prove the existence of nontrivial solutions of (1.3). Namely

Theorem 4.1. *Problem* (1.3) *has a nontrivial nonnegative solution.*

Proof. Let R_n, S_n be sequences such that $g(R_n) \neq 0$, $g(S_n) \neq 0$ and $\lim_{n\to\infty} R_n = \infty$, $\lim_{n\to\infty} S_n = -\infty$. Define $g_n(x) = g(x)$ if $x \in [S_n, R_n]$, $g_n(x) = g(R_n)$ if $x \in [R_n, \infty)$ and $g_n(x) = g(S_n)$ if $x \in (-\infty, S_n]$.

Denote by p_n the bounded solution of (1.3), with $g \equiv g_n$, provided by Lemma 4.1 satisfying

$$\int_{-b}^{b} p_n(t)\,dt = 1.$$

Fix $M > 0$ such that $g(M) > 0$ and $g(-M) > 0$. By Lemma 4.3 there exists a constant A, independent of n, such that

$$\max_{x\in[-M,M]} p_n \leqslant A \quad \text{for all } n.$$

Proceeding as in Lemma 2.2 this bound implies that $\{p_n\}_{n\in\mathbb{N}}$ restricted to $[-M, M]$ is equicontinuous. A standard diagonalization argument provides a subsequence, still denoted by p_n, which converges uniformly on compact subsets of $\mathbb{R}$ to a nontrivial continuous function p.

Letting $n \to \infty$ in the equation

$$p_n(x) = \int_{\mathbb{R}} J\left(\frac{x-y}{g_n(y)}\right) \frac{p_n(y)}{g_n(y)}\,dy,$$

we obtain that p solves (1.3) as desired. □

In the next result we show that a necessary condition to have bounded solutions of (1.3) is that $g(x)$ cannot converge to zero when $x \to \infty$ or $x \to -\infty$.

Theorem 4.2. *Suppose that $g(x) \to 0$ as $x \to \infty$ or $x \to -\infty$. Then all nontrivial nonnegative solutions of* (1.3) *are unbounded.*

Proof. We proceed by contradiction. Suppose that p is a nontrivial nonnegative bounded solution of (1.3) and, without loss of generality, $g(x) \to 0$ as $x \to \infty$.

Since p is nontrivial, it is easy to see that there exist $c_1 > 0$, $x_0 \in \mathbb{R}$ such that

$$\int_0^b \int_{x_0-w}^{x_0+w} p(s) \int_{\frac{w}{g(s)}}^{1} J(z)\,dz\,ds\,dw > c_1 > 0. \tag{4.15}$$

As $g(x) \to 0$ as $x \to \infty$ we have that for any $\delta > 0$ there exists $M > 0$ such that $g(x) < \delta$ for all $x \geqslant M$, thus if $x \geqslant M + \delta$ we have

$$\int_0^b \int_{x-w}^{x+w} p(s) \int_{\frac{w}{g(s)}}^{1} J(z)dz\,ds\,dw \leqslant \int_0^\delta \int_{x-\delta}^{x+\delta} p(s)\,ds\,dw \leqslant 2\|p\|_\infty \delta^2.$$

By virtue of Lemma 4.2, we contradict (4.15) taking $\delta \to 0$. $\square$

The following two theorems provide sufficient conditions on g that guarantee upper and lower bounds for the solutions of (1.3).

Theorem 4.3. *Assume that g satisfies* (g3) *and*

(g4) $\limsup_{x\to\infty} g(x) > 0$, $\limsup_{x\to-\infty} g(x) > 0$.

Then Eq. (1.3) *admits a positive bounded solution.*

Proof. By hypothesis there exist a constant $a_4 > 0$ and sequences $R_n \to \infty$ and $S_n \to -\infty$ such that $g(S_n) > a_4$ and $g(R_n) > a_4$ for all n. As in the proof of Theorem 4.1, we define $g_n(x) = g(x)$ if $x \in [S_n, R_n]$, $g_n(x) = g(R_n)$ if $x \in [R_n, \infty)$ and $g_n(x) = g(S_n)$ if $x \in (-\infty, S_n]$, and we let p_n be the bounded solution of (1.3) with $g \equiv g_n$ satisfying

$$\int_{-b}^{b} p_n(t)\,dt = 1.$$

Arguing exactly as in the proof of Theorem 4.1, the result will be proved if we show that there exists $C > 0$ such that $\|p_n\|_\infty \leqslant C$ for all n. To do this, choose $a < \min\{a_3, a_4\}$. For any $x_0 \in \mathbb{R}$ we have

$$p_n(x_0) = \int_{g_n(y)<a} J\left(\frac{x_0 - y}{g_n(y)}\right)\frac{p_n(y)}{g_n(y)}\,dy + \int_{g_n(y)\geqslant a} J\left(\frac{x_0 - y}{g_n(y)}\right)\frac{p_n(y)}{g_n(y)}\,dy,$$

therefore

$$p(x_0) \leqslant \|p_n\|_\infty \beta + \frac{\|J\|_\infty}{a} \int_{\substack{[x_0-b,\, x_0+b]\\ g(y)\geqslant a}} p_n(y)\,dy, \tag{4.16}$$

since $g_n(y) = g(y)$ whenever $g_n(y) \leqslant a$. Proceeding as in (4.11) we have that for any $x_1 \in \mathbb{R}$

$$\int_0^b \int_{-w}^{w} p_n(s) \int_{\frac{w}{g_n(s)}}^{1} J(z)\,dz\,ds\,dw \geqslant \frac{a}{4} \int_{\frac{1}{2}}^{1} J(z)\,dz \int_{\substack{[x_1-\frac{a}{4},\, x_1+\frac{a}{4}]\\ g(y)\geqslant a}} p_n(s)\,ds.$$

Since $\int_{-b}^{b} p_n(s)\,ds = 1$ and the interval $[x_0 - b, x_0 + b]$ can be covered by a finite number of intervals of the form $[x_1 - a, x_1 + a]$, we can use the above inequality and (4.16) to obtain

$$p_n(x_0) \leqslant \beta \|p_n\|_\infty + C(a, b) \quad \text{for all } n.$$

Recalling that $\beta < 1$, this inequality gives the desired result. □

Theorem 4.4. *Assume that g satisfies*

(g5) *there exist constants $0 < \gamma < 1$ and $C_5 > 0$ such that*

$$\bigl|\{x \in I \mid g(x) \leqslant a\}\bigr| \leqslant C_5 a^{\frac{1}{\gamma}}$$

for any interval I with $|I| \leqslant 2b$.

Then for any nonnegative nontrivial bounded solution p of (1.3) there exists $d > 0$ such that

$$p(x) \geqslant d \quad \textit{for all } x \in \mathbb{R}.$$

Remark 4.1. We observe that the hypothesis (g5) implies (g3) and (g4). Therefore if p is the solution of (1.3) constructed in Theorem 4.3 we have that $\|p\|_\infty < \infty$.

Proof of Theorem 4.4. By Lemma 4.2 there exists a constant $P > 0$ such that

$$\int_0^b \int_{D-b}^{D+b} p(s) \int_{\frac{w}{g(s)}} J(z)\,dz\,ds\,dw = P \quad \text{for all } D \in \mathbb{R}.$$

Hence for a fixed $0 < a_0 < b$ we have

$$\begin{aligned} P &= \int_0^b \int_{\substack{[D-b,\, D+b]\\ g(y)<a_0}} p(s) \int_{\frac{w}{g(s)}} J(z)\,dz\,ds\,dw + \int_0^b \int_{\substack{[D-b,\, D+b]\\ g(y)\geqslant a_0}} p(s) \int_{\frac{w}{g(s)}} J(z)\,dz\,ds\,dw \\ &= I_1 + I_2. \end{aligned}$$

If $g(s) < a_0$ and $w \geqslant a_0$ then $w/g(s) > 1$ from where we obtain

$$I_1 \leqslant \int_0^{a_0} \int_{D-a_0}^{D+a_0} p(s)\, ds\, dw \leqslant 2a_0^2 \|p\|_\infty.$$

Therefore if $a_0 \leqslant (\frac{P}{4\|p\|_\infty})^{1/2}$ we have

$$\int_{\substack{[D-b,\, D+b] \\ g(y) \geqslant a_0}} p(s)\, ds \geqslant \frac{P}{2b}. \tag{4.17}$$

On the other hand, if $x_1 \in \mathbb{R}$ and $a_1 > 0$ then

$$\begin{aligned} p(x_1) &\geqslant \int_{g(y) \geqslant a_1} J\left(\frac{x_1 - y}{g(y)}\right) \frac{p(y)}{g(y)}\, dy \\ &\geqslant \int_{\substack{[x_1 - \frac{a_1}{2},\, x_1 + \frac{a_1}{2}] \\ g(y) \geqslant a_1}} J\left(\frac{x_1 - y}{g(y)}\right) \frac{p(y)}{g(y)}\, dy \\ &\geqslant \frac{m}{b} \int_{\substack{[x_1 - \frac{a_1}{2},\, x_1 + \frac{a_1}{2}] \\ g(y) \geqslant a_1}} p(y)\, dy, \end{aligned}$$

where $m = \min_{|z| \leqslant 1/2} J(z)$. Thus we obtain

$$\int_{\substack{[x_1 - \frac{a_1}{2},\, x_1 + \frac{a_1}{2}] \\ g(y) \geqslant a_1}} p(y)\, dy \leqslant \frac{b}{m}\, p(x_1), \tag{4.18}$$

from where in particular

$$\int_{\substack{[x_1 + \frac{a_1}{4},\, x_1 + \frac{a_1}{2}] \\ g(y) \geqslant a_1}} p(y)\, dy \leqslant \frac{b}{m}\, p(x_1). \tag{4.19}$$

By hypothesis (g5) we have that

$$\left| \left\{ x \in \left[x_1 + \frac{a_1}{4}, x_1 + \frac{a_1}{2} \right] \,\middle|\, g(y) < a_1 \right\} \right| \leqslant C_5 a_1^{1/\gamma},$$

thus, we can choose a_1 such that

$$C_5 a_1^{1/\gamma} \leqslant \frac{a_1}{8}, \quad a_1 \leqslant \left(\frac{P}{4\|p\|_\infty} \right)^{1/2},$$

for which

$$\left|\left\{x \in \left[x_1 + \frac{a_1}{4}, x_1 + \frac{a_1}{2}\right] \;\middle|\; g(y) \geqslant a_1\right\}\right| \geqslant \frac{a_1}{8},$$

and then by (4.19) there exists $x_2 \in [x_1 + a/4, x_1 + a/2]$ with

$$p(x_2) \leqslant \frac{b}{m}\frac{8}{a_1}p(x_1).$$

Repeating the above procedure with $p(x_2)$ instead of $p(x_1)$ we obtain

$$\int\limits_{\substack{[x_2-\frac{a_1}{2},\, x_2+\frac{a_1}{2}]\\ g(y)\geqslant a_1}} p(y)\,dy \leqslant \frac{b}{m}p(x_2) \leqslant \left(\frac{b}{m}\right)^2 \frac{8}{a_1}\,p(x_1).$$

As $x_1 + a_1/4 \leqslant x_2 \leqslant x_1 + a_1/2$ from the above inequality we have

$$\int\limits_{\substack{[x_1,\, x_2+\frac{3a_1}{4}]\\ g(y)\geqslant a_1}} p(y)\,dy \leqslant \left(\frac{b}{m}\right)^2 \frac{8}{a_1}p(x_1),$$

and then from (4.18)

$$\int\limits_{\substack{[x_1-\frac{a_1}{2},\, x_1+\frac{3a_1}{4}]\\ g(y)\geqslant a_1}} p(y)\,dy \leqslant p(x_1)\left[\left(\frac{b}{m}\right)^2 \frac{8}{a_1} + \frac{b}{m}\right].$$

Since a_1 is fixed, we can use the same procedure a finite number of times to show that there exists a positive constant $C(b, m, a_1)$ such that

$$\int\limits_{\substack{[x_1-b,\, x_1+b]\\ g(y)\geqslant a_1}} p(y)\,dy \leqslant p(x_1)C(b, m, a_1),$$

and then using (4.17) we conclude that

$$p(x_1) \geqslant \frac{P}{2bC(b, m, a_1)}. \qquad \square$$

5. Asymptotic behavior

In this section we study the asymptotic behavior of solutions of (1.2) under the additional assumption that (1.3) possesses a solution p such that $p \geqslant c$ in $\mathbb{R}$ for some $c > 0$. Observe that by Theorem 4.4 hypothesis (g5) implies the existence of such a p. Throughout this section we shall assume that such a solution exists and it will be denoted by p.

An important tool that will be used is a Lyapunov functional, that is defined following the ideas introduced by Michel, Mischler and Perthame in [12] in their study of the asymptotic behavior of solutions of some linear fragmentation-growth models using a relative entropy inequality.

Theorem 5.1. *Let u be a solution of* (1.2) *with initial value $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then the following identity holds*:

$$E'(t) = -\int_{\mathbb{R}}\int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\left[\frac{u}{p}(t,x) - \frac{u}{p}(t,y)\right]^2 dy\,dx, \tag{5.1}$$

where

$$E(t) = \int_{\mathbb{R}} \frac{u^2}{p}\,dx. \tag{5.2}$$

Proof. Under our assumptions E is well defined and differentiable. Moreover, its derivative is given by

$$E'(t) = 2\int_{\mathbb{R}}\int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)}{g(y)}\frac{u(x,t)}{p(x)}\,dy\,dx - 2\int_{\mathbb{R}} \frac{u^2(x,t)}{p(x)}\,dx. \tag{5.3}$$

Using that p is a solution of (1.3) we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\frac{u^2(x,t)}{p^2(x)}\,dy\,dx = \int_{\mathbb{R}} \frac{u^2(x,t)}{p(x)}\,dx,$$

and

$$\int_{\mathbb{R}}\int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\frac{u^2(y,t)}{p^2(y)}\,dy\,dx = \int_{\mathbb{R}} \frac{u^2(y,t)}{p(y)}\,dy,$$

we obtain from (5.3) the desired result. □

Let us now prove some regularity properties of this energy.

Lemma 5.1. *Suppose that the hypothesis of Theorem* 5.1 *holds. Then $E(t) \in C^{1,1}(\mathbb{R}^+)$.*

Proof. Let t_1 and t_2 be in $\mathbb{R}^+$. Using formula (5.1) we have

$$\left|E'(t_1) - E'(t_2)\right| \leqslant \int_{\mathbb{R}^2} J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\Gamma(t_1,t_2,x,y)\,dy\,dx, \tag{5.4}$$

where

$$\Gamma(t_1,t_2,x,y) = \left|\left[\frac{u}{p}(y,t_2) - \frac{u}{p}(x,t_2)\right]^2 - \left[\frac{u}{p}(y,t_1) - \frac{u}{p}(x,t_1)\right]^2\right|.$$

By Proposition 3.3 the function $u(x,t)$ is Lipschitz in time uniformly in x, thus there exists a constant κ such that

$$\Gamma(t_1,t_2,x,y) \leqslant 2\kappa|t_1-t_2|\left(\frac{|u|}{p}(y,t_2)+\frac{|u|}{p}(x,t_2)+\frac{|u|}{p}(y,t_1)+\frac{|u|}{p}(x,t_1)\right).$$

Observing that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\frac{|u(y,t)|}{v(y)}\,dy\,dx = \int_{\mathbb{R}}\left|u(x,t)\right|dx \leqslant \int_{\mathbb{R}}\left|u_0(x)\right|dx,$$

and

$$\int_{\mathbb{R}}\int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\frac{|u(x,t)|}{v(x)}\,dy\,dx = \int_{\mathbb{R}}\left|u(x,t)\right|dx \leqslant \int_{\mathbb{R}}\left|u_0(x)\right|dx,$$

we deduce from (5.4) that

$$\left|E'(t_1)-E'(t_2)\right| \leqslant 4C|t_1-t_2|\int_{\mathbb{R}} u_0(x)\,dx. \qquad \square$$

Before giving the result concerning the asymptotic behavior of the solutions of (1.2), we first prove a technical lemma.

Lemma 5.2. *Suppose that $w \in L^2(\mathbb{R})$ satisfies*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{1}{g(y)}\big[w(x)-w(y)\big]^2\,dy\,dx = 0, \tag{5.5}$$

then there exists $\lambda \in \mathbb{R}$ such that $w(x)=\lambda$ a.e.

Proof. If (5.5) holds we have

$$J\left(\frac{x-y}{g(y)}\right)\frac{1}{g(y)}\big[w(x)-w(y)\big]^2 = 0 \quad \text{a.e. in } \mathbb{R}^2. \tag{5.6}$$

Let I be an open interval where $g>0$. We claim that there exists λ such that $w(x)=\lambda$ a.e. in I. Indeed, let $D=\{(x,x) \mid x\in I\}$ note that there exist sequences $\{x_i\}_{i\in\mathbb{Z}}$ and $\{\delta_i\}_{i\in\mathbb{Z}}$ such that $x_i<x_{i+1}<x_i+\delta_i$,

$$J\left(\frac{x-y}{g(y)}\right)\frac{1}{g(y)} > 0 \quad \text{in } R_i, \tag{5.7}$$

where $R_i=[x_i-\delta_i, x_i+\delta_i]^2$, and $D\subset\bigcup_{i\in\mathbb{Z}} R_i$. By (5.7) and (5.6) we have that

$$w(x)-w(y)=0 \quad \text{a.e. in } R_i,$$

thus, for each $i \in \mathbb{Z}$ there exists λ_i such that $w(x) = \lambda_i$ in $[x_i - \delta_i, x_i + \delta_i]$. Since in the interval $(x_{i+1} - \delta_{i+1}, x_i + \delta_i)$ we have

$$\lambda_{i+1} = w(x) = \lambda_i \quad \text{a.e.},$$

the claim is proved.

Let $I_1 = (z_1, z_2)$ and $I_2 = (z_2, z_3)$ be two open intervals with $g > 0$ in $I_1 \cup I_2$ and $g(z_2) = 0$. By the claim there exist λ_1, λ_2 such that $w(x) = \lambda_i$ a.e. in I_i for $i = 1, 2$. The result will be proved if we show that $\lambda_1 = \lambda_2$.

By (g2) there exist positive constants $C, r > 0$ and $\alpha < 1$ such that $g(y) \geqslant C|y - z_2|^a$ for all $y \in [z_2 - r, z_2 + r]$. We set $0 < r_0 < \min\{(r/2)^a C/2, r, (C/2)^{1/1-\alpha}\}$ and $z_2 - r_0 < x < z_2$. If $y \in I_2$ satisfies

$$\left(\frac{2(z_2 - x)}{C}\right)^{1/\alpha} < y - z_2 < r_0,$$

then

$$z_2 - x < C\frac{(y - z_2)^\alpha}{2},$$

and using $r_0 < (C/2)^{1/1-\alpha}$ we obtain

$$C\frac{(y - z_2)^\alpha}{2} < C(y - z_2)^\alpha - (y - z_2) < g(y) - (y - z_2).$$

Hence, from the above inequalities, we have $y - x < g(y)$. Therefore, there exists $\eta > 0$, $\tilde{x} \in I_1$ and $\tilde{y} \in I_2$ such that

$$w(x) - w(y) = 0 \quad \text{a.e. for } (x, y) \in [\tilde{x} - \eta, \tilde{x} + \eta] \times [\tilde{y} - \eta, \tilde{y} + \eta].$$

Since $w(x) = \lambda_1$ a.e. in I_1 and $w(y) = \lambda_2$ a.e. in I_2 we have $\lambda_1 = \lambda_2$. $\square$

We are now in position to prove the results about the asymptotic behavior of the solutions of (1.2).

Theorem 5.2. *Assume $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then $u \to 0$ weakly in $L^2(\mathbb{R})$ as $t \to \infty$.*

Proof. Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence such that $t_n \to +\infty$. We define the sequence of functions $\{u_n\}_{n\in\mathbb{N}}$ by $u_n(x) \equiv u(x, t_n)$. From Proposition 3.3 the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R})$, therefore a subsequence, which we still call $\{u_n\}$, converges weakly in $L^2(\mathbb{R})$ to some $\bar{u}$.

Using Lemma 5.1 and the monotonicity of $E(t)$ we see that

$$E'(t) \to 0 \quad \text{as } t \to \infty,$$

hence

$$E'(t_n) = \int_{\mathbb{R}} \int_{\mathbb{R}} J\left(\frac{x - y}{g(y)}\right) \frac{p(y)}{g(y)} \left[\frac{u_n}{p}(x) - \frac{u_n}{p}(y)\right]^2 dy\, dx \to 0.$$

Let $\Pi_R := [-R, R]^2$ for $R > 0$, then

$$\begin{aligned}
&\int\limits_{\Pi_R} J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\left[\frac{\bar{u}}{p}(x)-\frac{\bar{u}}{p}(y)\right]^2 dy\,dx \\
&\quad\leqslant \liminf_{n\to\infty}\int\limits_{\Pi_R} J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\left[\frac{u_n}{p}(x)-\frac{u_n}{p}(y)\right]^2 dy\,dx \\
&\quad\leqslant \liminf_{n\to\infty}\int\limits_{\mathbb{R}^2} J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\left[\frac{u_n}{p}(x)-\frac{u_n}{p}(y)\right]^2 dy\,dx \\
&\quad\leqslant 0.
\end{aligned}$$

Whence, we have

$$\lim_{\mathbb{R}\to+\infty}\int\limits_{\Pi_R} J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\left[\frac{\bar{u}}{p}(x)-\frac{\bar{u}}{p}(y)\right]^2 dy\,dx \leqslant 0,$$

which implies that

$$\int\limits_{\mathbb{R}^2} J\left(\frac{x-y}{g(y)}\right)\frac{p(y)}{g(y)}\left[\frac{\bar{u}}{p}(x)-\frac{\bar{u}}{p}(y)\right]^2 dy\,dx = 0.$$

Using Lemma 5.2 we have that $\bar{u} = \lambda p$ for some λ. Since $\bar{u}$ is in $L^2(\mathbb{R})$ and p is bounded from below, we conclude that $\lambda = 0$, that is $\bar{u} \equiv 0$. It follows that $u(\cdot, t)$ converges weakly to 0 in $L^2(\mathbb{R})$ as $t \to \infty$. $\square$

A consequence of Theorem 5.2 is the following result.

Theorem 5.3. *Assume $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then $u(\cdot, t) \to 0$ in $L^q_{\mathrm{loc}}(\mathbb{R})$ for any $1 \leqslant q \leqslant \infty$, as $t \to \infty$.*

Proof. We first consider $1 \leqslant q < \infty$. Let Ω be a compact subset of $\mathbb{R}$. Using Proposition 3.3 and Theorem 5.2 we see that

$$0 \leqslant \int\limits_{\Omega} u^q(t, x) \leqslant K_q \int\limits_{\mathbb{R}} u(t, x) 1_\Omega(x)\, dx \to 0. \tag{5.8}$$

Consider now $q = \infty$. By hypothesis (g2), given $[-K, K]$ there exists $r > 1$ and $M > 0$ such that for all $x \in [-K, K]$

$$\left\| J\left(\frac{x-\cdot}{g(\cdot)}\right)\frac{1}{g(\cdot)}\right\|_{L^r[-K-b,K+b]} \leqslant M.$$

Define r^* such that $1/r + 1/r^* = 1$. By (5.8), given $\varepsilon > 0$ there exists a $t_0 > 0$ such that

$$\|u(\cdot,t)\|_{L^{r^*}[-K-b,K+b]} \leqslant \varepsilon \quad \text{for } t \geqslant t_0.$$

So if $x \in [-K, K]$ and $t \geqslant t_0$, we have

$$\begin{aligned} u(x,t) &= e^{-(t-t_0)}u(x,t_0) + \int_{t_0}^{t} e^{-(t-s)} \int_{\mathbb{R}} J\left(\frac{x-y}{g(y)}\right)\frac{u(y,s)}{g(y)}\,dy\,ds \\ &\leqslant e^{-(t-t_0)}u(x,t_0) + M\varepsilon. \end{aligned}$$

From where $\|u(\cdot,t)\|_{L^\infty[-K,K]} \to 0$ as $t \to \infty$. $\square$

When $u_0 \in L^1(\mathbb{R})$ but not in $L^\infty(\mathbb{R})$ we still have the following convergence result.

Theorem 5.4. *Assume $u_0 \in L^1(\mathbb{R})$. Then $u(\cdot,t) \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R})$.*

Proof. Set Ω a compact subset of $\mathbb{R}$ and $\varepsilon > 0$. We decompose $u_0 = w_1 + w_2$ with w_1, w_2, $w_1 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\|w_2\|_{L^1(\mathbb{R})} \leqslant \varepsilon$. By Theorem 5.3 and linearity we have the result. $\square$

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